9/22/25

Lecture 13

<u>Def</u> - A matrix A is <u>invertible</u> if $\exists$ matrix B such that $AB = I$ and $BA = I$.
We then call B the <u>inverse</u> of A and write $A^{-1} = B$.

<u>Theorem</u>: If A is an invertible $m \times n$ matrix, then $\forall \vec{b} \in \mathbb{R}^m$ $\exists$ unique $\vec{x} \in \mathbb{R}^n$ such that $A\vec{x} = \vec{b}$.

$\boxed{\text{e.g.}}$ Not every matrix is invertible

$$A = \begin{bmatrix} 2 & 4 \\ 3 & 6 \end{bmatrix} \quad \text{is not invertible}$$

To find A^{-1}, solve $\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 3 & 6 & 0 & 1 \end{array}\right] \to \dots \to \left[I \mid A^{-1} \right]$

$$\left[\begin{array}{cc|cc} 2 & 4 & 1 & 0 \\ 3 & 6 & 0 & 1 \end{array}\right] \begin{array}{l} \cdot \frac{1}{2} \\ \cdot \frac{1}{3} \end{array}$$

$$\to \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 1 & 2 & 0 & \frac{1}{3} \end{array}\right] \begin{array}{l} \\ -R_1 \end{array}$$

$$\to \left[\begin{array}{cc|cc} 1 & 2 & \frac{1}{2} & 0 \\ 0 & 0 & -\frac{1}{2} & \frac{1}{3} \end{array}\right]$$

left side is not row equivalent to I, so A is not invertible.

<u>Thm</u>: If A is 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, then A is invertible i.f.f. $ad - bc \neq 0$.
In this case: $A^{-1} = \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Cor - Invertible matrices are square. (ALL of them)

Proof: If A is invertible then $A\vec{x} = \vec{b}$ always has exactly one solution. Hence A is row equivalent to a RREF matrix $\tilde{A}$ with a pivot in each row. Since $A\vec{x} = \vec{b}$ never has infinetly many solutions, $\tilde{A}$ has a pivot in each column to.
Since each row and column of $\tilde{A}$ have at most one pivot, I see that $\tilde{A}$ of A has same number of rows and columns, i.e. A is square. ■

Theorem: The Following Are Equivalent (TFAE) for an $n \times n$ matrix of A.

1. A is invertible
2. A is row equivalent to I
3. A has rank n
4. $A\vec{x} = \vec{b}$ always has one solution
5. $\exists$ B such that $AB = I$
6. $\exists$ B such that $BA = I$

- Suppose $A = n \times n$ we find a matrix B such that $BA = I$

  I claim that then A is invertible. To see this consider a linear system:

$$A\vec{x} = \vec{b}$$
$$B(A\vec{x}) = B\vec{b}$$
$$(BA)\vec{x} = B\vec{b}$$
$$\vec{x} = B\vec{b}$$

- This shows $A\vec{x} = \vec{b}$ has at most one solution.

- Suppose A and B are invertible $n \times n$ matrices
  - Then
    - A^{-1} is invertible with $(A^{-1})^{-1} = A$
    - $\forall$ positive integer n, A^n is invertible with $(A^n)^{-1} = (A^{-1})^n$
    - AB is invertible with $(AB)^{-1} = B^{-1}A^{-1}$

↳ Check $(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1}$

$$= (AI)A^{-1}$$

$$= AA^{-1}$$

$$= I$$

$\boxed{Ex}$ $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$, $B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, are both invertible

$$\to \left(\begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \right)^{-1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}^{-1} \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}^{-1}$$

$$= \frac{1}{-1} \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$ } use the formula for the inverse of 2x2 matrices above.

$$= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -5 \\ -1 & 2 \end{bmatrix}$$

<u>Def</u> - A f_n $T : \mathbb{R}^T \to \mathbb{R}^m$ is <u>invertible</u> if $\exists$ another f_n $S : \mathbb{R}^m \to \mathbb{R}^n$ such that $S \circ T(\vec{x}) = \vec{x}$ and $T \circ S(\vec{y}) = \vec{y}$ $\forall$ $\vec{x} \in \mathbb{R}^n$ and $\vec{y} \in \mathbb{R}^m$. We write $T^{-1} = S$.

<u>Theorem</u>: If $T : \mathbb{R}^n \to \mathbb{R}^m$ is an invertible linear transformation, then $T^{-1} : \mathbb{R}^m \to \mathbb{R}^n$ is also a linear transformation. Hence $[T^{-1}] = [T]^{-1}$.